Differential Equations of the first order but not of the first degree.

Introduction:→

An equation of first order and nth degree may be written as

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots\cdots + P_{n-1} p + P_n = 0 \quad -(1)$$

Where $p = \frac{dy}{dx}$ and $P_1, P_2, \cdots, P_n$ are given functions of x & y.

We discuss three special types of such equations:

1. Those solvable for p.
2. Those solvable for y.
3. Those solvable for x.

1. Equations solvable for p :-

If we ~~took~~ look upon equation (1) as an algebracial equation in p, which has n roots $p_1, p_2, \cdots, p_n$, these being functions of x & y, we may write (1) in the form

$$(p-p_1)(p-p_2)\cdots\cdots(p-p_n) = 0 \quad — \quad (2)$$

Let $\phi_r(x, y, c_r) = 0$ be the ~~pre~~ primitives of $p - p_r = 0$, $r = 1, 2, 3, \cdots, n$.

All posible solutions of the equation (1) will then be included in the relation

$$\phi_1(x,y,c_1)\cdot\phi_2(x,y,c_2)\cdots\cdots\phi_n(x,y,c_n)=0 \quad \text{—} \quad (3)$$

$\because$ The general solution apparently contains n arbitrary constants, whereas we expect only one, as the equation is only first order. See that the generality will still be maintained if all the constants $c_1, c_2, \cdots, c_n$ be made the same, say c; may have any numerical value for a particular solution of the equation (1) is obtained by solving one or other of the equations $\phi_r(x,y,c)=0$, c assuming any arbitrary value.

Hence the complete primitive of (1) is the product

$$\phi_1(x,y,c)\cdot\phi_2(x,y,c)\cdots\cdots\phi_n(x,y,c)=0$$

Where c is any arbitrary constant.

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Example ① Solve the equation

$$p^2 + px + py + xy = 0 \quad \text{—— ①}$$

Solution :→

$\because \quad p^2 + px + py + x \cdot y = 0$

$\Rightarrow \quad p(p+x) + y(p+x) = 0$

$\Rightarrow \quad (p+x)(p+y) = 0 \quad \text{—— ②}$

From ②

If $p + x = 0 \Rightarrow \dfrac{dy}{dx} + x = 0$

$\Rightarrow \dfrac{dy}{dx} = -x$

$\Rightarrow dy = -x\,dx$

integrating, we have

$y = -\dfrac{x^2}{2} + c_1'$

$\Rightarrow 2y = -x^2 + c_1$

& $p + y = 0 \Rightarrow \dfrac{dy}{dx} = -y$

$\Rightarrow \dfrac{dy}{y} = -dx$

integrating, we have

$x = -\log y + c_2$

The complete primitive can be written as

$(2y + x^2 - c)(x + \log y - c) = 0$

Where c is an arbitrary constant.

Example (2) Solve the equation

$$x^2p^2 - 2xyp + y^2 = x^2y^2 + x^4 \quad \text{--- (1)}$$

Solⁿ:→ Equation (1) can be written as

$$(xp-y)^2 = x^2(x^2+y^2)$$

$$\Rightarrow xp - y = \pm x\sqrt{x^2+y^2}$$

$$\Rightarrow p = \frac{y \pm x\sqrt{x^2+y^2}}{x}$$

putting $y = vx$, we see that

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx \pm x\sqrt{x^2+x^2v^2}}{x}$$

$$\Rightarrow \frac{dv}{dx} = \pm\sqrt{1+v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \pm dx$$

& Hence $v = \sinh(x+c)$ & $v = \sinh(x-c)$

∴ The general solution is

$$\{y - x\sinh(x+c)\}\{y - x\sinh(c-x)\} = 0$$

Where c is an arbitrary constant.

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